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COMPARATIVE STUDY OF SOME ITERATIVE METHODS FOR NONLINEAR EQUATIONS FROM A DYNAMICAL POINT OF VIEW

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ABSTRACT

In this paper, we investigate and compare several optimal fourth and eighth-order iterative methods for solving nonlinear equations, examining their basins of attraction through lower and higher-degree polynomials. The programming package MATLAB is used to plot basins of attraction for each of the iterative method. Through dynamical analysis using basins of attraction, we identify the most stable and effective optimal eighth-order method, which has wider region of convergence in comparison to existing root-finding methods of similar nature.

KEYWORDS: Iterative methods, Nonlinear Equations, Basins of attraction, Optimal order of convergence, Simple roots

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1. INTRODUCTION

The problem of solving nonlinear equations has garnered significant interest across various domains such as science, mathematics, and numerous real-world phenomena, including weather forecasting, precise satellite positioning in designated orbits, measuring earthquake magnitudes, etc. Various methods, such as iterative methods like Newton-type methods and Steffensen-type methods, have been developed to tackle nonlinear equations efficiently. These methods play a pivotal role in diverse applications, including physics, biology, economics, and engineering. Whether it's predicting the trajectory of celestial bodies, optimizing chemical processes, or analyzing the dynamics of biological systems, the ability to solve nonlinear equations empowers researchers to unravel the complexities of the world around us and devise effective solutions to a wide range of problems. We can write a nonlinear equation as $h(z) = 0$ for a complex function, $h: \Psi \subset \mathbb{C} \to \mathbb{C}$ where $z \in C$. Iterative approaches, that start with an initial guess, are used to provide approximate solutions to nonlinear equations [1]. Iterative techniques are compared according to order of convergence, O, computations of function or function evaluations, d, and the Ostrowski's efficiency index, $E_f = O^{\frac{1}{d}}$ [2] . Kung and Traub [3] introduced the notion of optimal iterative techniques as a multistep iterative scheme (without memory) using $d+1$ function evaluations has maximum order of convergence of 2^d.

In this paper, we have study a better technique, known as basins of attractions, to compare the iterative root finding methods visually in terms of plotting their regions of convergence. Basins of attraction refer to regions in the domain of a problem where different initial guesses lead to convergence towards specific attractors, such as fixed points and roots of nonlinear equations. In the context of root-finding methods for solving nonlinear equations, basins of attraction illustrate how different initial guesses influence the convergence behaviour of iterative methods. By plotting these basins, one can visually depict which initial guesses lead to convergence towards a specific root and which ones diverge. Analysing basins of attraction helps in understanding the robustness and efficiency of root-finding methods and aids in selecting appropriate starting points to achieve fast convergence.

Basins of attraction serve as a visual tool to compare and analyse the performance of different root-finding methods. With the help of drawing the plots of basins of attraction of various iterative methods, we gain insight into how each method behaves for different problems for given sent of initial guesses [21]. This comparative analysis allows us to identify the stability, convergence properties, and efficiency of each method across different initial guesses. Understanding the basins of attraction aids in selecting the most suitable method for a given problem, considering factors such as robustness and convergence speed. Through this approach, researchers can make informed decisions to optimize the choice of root-finding methods for their specific application, ultimately enhancing the efficacy of numerical solutions for nonlinear equations. An iterative method is better if it has a wide area of convergence. The procedure is such that it assigns n colors to the n basins, apply a root finding method to obtain which initial guesses converge to a specific basin of the root within an interval, and that initial guesses are painted with a color already selected for the relevant root. The concept of plotting the basins of attraction was firstly given by Stewart [4] (2001). After that several researchers have studied basins of attaction for root finding methods, for exmple, Varona [22], Amat et al. [5] (2005), Scott et al. [6] (2011), Chun et al. [7] (2012), Chicharro et al. [8] (2013), Neta et al. [9] (2014), Zafar et al. [10] (2015), Junjua et al. [11] (2015), Chun et al. [12] (2016), Daza [23] (2022), Varona [25] (2022).

Inspired by the research on this track and with a need to obtain most efficient and reliable root finding methods, in this paper, we compare various iterative methods from a dynamical viewpoint. There are several higher order iterative methods for solving nonlinear equations in literature, only few of them are optimal and use less computational cost. Moreover, each method behaves differently in terms of error and convergence regions for different nonlinear problems. Therefore, we have taken some well-known optimal fourth and eighth order iterative schemes for the comparison and to find out a best iterative technique among them. We describe various iterative methods of optimal order four and eight for the comparison in Section 2. In Section 3, we compare these methods by drawing the plots of their basins of attraction in the complex plane. The Section 4 provides the conclusions.

2. ITERATIVE METHODS FOR COMPARISON

In this section, we write some iterative root finding methods of optimal order four and eight for the sake of their comparison as follows.

- Super Halley (SH4) method of optimal order four
- Modified super Halley (MSH4) method of optimal order four
- King's method of optimal order four (K4)
- Jarratt's (J4) method of optimal order four
- Kung-Traub (HKT8) method of optimal order eight based on Hermite formula
- Neta's (N8) method of optimal order eight
- Wang-Liu's (WL8) method of optimal order eight based on Hermite formula
- Optimal eighth order method of Sharma and Arora (SA8)
- Behl's (RM8) method of optimal order eight without derivatives
- Sivakumar's (SK8) method of optimal order eight

The iteration formulae of all of the above methods are given as follows.

1. Super Halley method, an optimal fourth order method, is given by (SH4) [13]:

$$y_k = z_k - \frac{2}{3}u_k,$$

$$z_{k+1} = z_k - \left(1 + \frac{1}{2}\frac{L_h}{1-L_h}\right)u_k, \tag{1}$$

where $L_h = \frac{h(z_k)h''(z_k)}{(h'(z_k))^2}$, and $u_k = \frac{h(z_k)}{h'(z_k)}$.

2. Modified super-Halley method of optimal order four, is of the following form (MSH4) [14]:

$$y_k = z_k - \frac{2}{3}u_k,$$

$$z_{k+1} = z_k - \left(1 + \frac{1}{2}\frac{\hat{L}_h}{1-\hat{L}_h}\right)u_k. \tag{2}$$

where $\hat{L}_h = \frac{h(z_k)}{(h'(z_k))^2}\frac{h'(y_k)-h'(z_k)}{y_k - z_k}$.

3. King's method of optimal order four is given as follows (K4) [15]:

$$y_k = z_k - \frac{v(z_k)}{v'(z_k)},$$

$$z_{k+1} = y_k - \frac{v(y_k)}{v'(z_k)}\frac{v(z_k)+\beta v(y_k)}{v(z_k)+(\beta-2)v(y_k)}. \tag{3}$$

3. Jarratt's optimal fourth order method is given as (J4) [16]:

$$y_k = z_k - \frac{2}{3}u_k,$$

$$z_{k+1} = z_k - \frac{1}{2}u_k - \frac{1}{2}\frac{u_k}{[1+\frac{3}{2}(\frac{h'(y_k)}{h'(z_k)}-1)]}. \tag{4}$$

4. Kung-Traub method of optimal order eight (HKT8) based on Hermite formula is given as [17]:

$$y_k = z_k - u_k,$$

$$w_k = y_k - \frac{h(y_k)}{h'(z_k)} \cdot \frac{1}{(1-\frac{h(y_k)}{h(z_k)})^2}, \tag{5}$$

$$w_{k+1} = w_k - \frac{h(w_k)}{H_3'(w_k)}.$$

where,

$$H'_3(x_k) = 2(h[w_k, x_k] - h[w_k, y_k]) + h[y_k, x_k] + \frac{y_k - x_k}{y_k - w_k}(h[w_k, y_k] - h'(w_k)).$$

5. Neta proposed an eighth order optimal method, which is given as (N8) [18]:

$$y_k = z_k - u_k,$$

$$x_k = y_k - \frac{h(y_k)}{h'(z_k)}\frac{h(z_k)+\beta h(y_k)}{h(z_k)+(\beta-2)h(y_k)}, \tag{6}$$

$$z_{k+1} = z_k - u_k + \gamma h^2(z_k) - \rho h^3(z_k).$$

Where we have,

$$\rho = \frac{\Psi_y - \Psi_x}{F_y - F_x}, \gamma = \Psi_y - \rho F_y, F_y = h(y_k) - h(z_k),$$

$$F_x = h(x_k) - h(z_k), \Psi_y = \frac{y_k - z_k}{F_y^2} - \frac{1}{F_y h'(z_k)},$$

$$\Psi_x = \frac{x_k - z_k}{F_x^2} - \frac{1}{F_x h'(z_k)}.$$

6. Optimal eighth-order method of Wang and Liu based on the Hermite interpolating polynomial is given below (WL8) [19]:

$$y_k = z_k - u_k,$$

$$x_k = y_k - \frac{h(y_k)}{h'(z_k)}\frac{h(z_k)}{h(z_k)-2h(y_k)}, \quad (7)$$

$$z_{k+1} = x_k - \frac{h(x_k)}{H_3'(x_k)}.$$

where $H_3'(x_k)$ is given as below:

$$H_3'(x_k) = 2(h[z_k,x_k] - h[z_k,y_k]) + h[y_k,x_k] + \frac{y_k - x_k}{y_k - z_k}(h[x_k,y_k] - h'(z_k)).$$

7. Optimal eighth-order method due to Sharma and Arora (SA8) is given by [20]:

$$y_k = z_k - \frac{h(z_k)}{h'(z_k)}$$

$$x_k = \Psi_4(z_k,y_k), \quad (8)$$

$$z_{k+1} = x_k - \frac{h[x_k,y_k]}{h[x_k,z_k]}\frac{h(x_k)}{2h[x_k,y_k]-h[x_k,z_k]}, \quad (9)$$

where,

$$\Psi_4(z_k,y_k) = y_k - \frac{h(y_k)}{2h[y_k,z_k]-h'(z_k)}. \quad (10)$$

8. Behl et al. [21] gave an optimal eighth-order iterative method:

$$y_k = w_k - -\frac{v(w_k)}{v'(w_k)},$$

$$x_k = y_k - \frac{v(y_k)}{v'(w_k)}(1 + 2u + 5u^2), \quad (11)$$

$$w_{k+1} = x_k - \frac{v(x_k)}{v'(w_k)}(1 + 2u + t + 6u^2 + 4ut + t^2 + 6u^3 + 14u^2t),$$

Where

$$u_k = \frac{v(y_k)}{v(w_k)},\ n_k = \frac{v(w_k)}{v\prime(w_k)},\ t_k = \frac{v(x_k)}{v\prime(y_k)}.$$

9. Sivakumar et al. [22] presented an optimal eighth-order iterative scheme given as:

$$y_k = w_k - \frac{j(w_k)}{j'(w_k)},$$

$$x_k = w_k - \frac{j(w_k)}{j'(w_k)}\left(\frac{j(w_k) - j(w_k)}{j(w_k) - 2j(w_k)}\right),$$

$$w_{k+1} = x_k - \frac{j(x_k)}{q'(x_k)},$$

Where

$$q'(x_k) = b_1 + 2b_2(x_k - w_k) + 3b_3(x_k - w_k)^2,\ \ b_1 = j'(w_k),$$

$$b_2 = \frac{j[y_k,w_k,w_k](x_k - w_k) - j[x_k,w_k,w_k](y_k - w_k)}{x_k - y_k},$$

$$b_3 = \frac{j[x_k,w_k,w_k] - j[y_k,w_k,w_k]}{x_k - y_k}.$$

3. DYNAMICAL COMPARISON OF ITERATIVE METHODS

This section is devoted to the dynamical comparison of iterative methods discussed in Section 2 by drawing their basins of attractions. The dynamical behaviour of optimal methods of order four and eight is presented on five different polynomials of degree 2, 3, 4, 5 and 6. We have drawn the plots of the basins of attractions for each method in complex plane. We have taken a square of $[-3,3]$ by $[-3,3]$ which contains all the roots of the polynomials considered. There are 360,000 equally distributed initial guesses to run each method in this square. Initial points that converge to any of the roots are coloured based on the corresponding root, whereas black color indicates the points that fail to converge within the specified criteria, meaning that after 40 iterations, the method does not converge the vicinity of any of the roots by a distance closer than 10^{-5}. The basins are drawn such that the fewer the number of iterations the brighter the color. Hence, ideally, the best method should exhibit lighter tones and less black points.

Example 1. First, we take the following quadratic polynomial:

$$p_2(z) = z^2 - 1,$$

with the roots at $+1$ and -1. In Figures 1, basins of attraction of SH4, MSH4, K4, J4, HKT8, N8, WL8, SA8, RM8, and SK8 are shown for p_2. For this example, the basins are plotted in Figure 1. From Figure 1, we conclude that SA8 stands out as the best and robust method since it has wide region of convergence and has brighter colors.

Example 2. We now take a cubic polynomial as follows:

$$p_3(z) = z^3 - z.$$

The above cubic equation has three roots at 0. -1 and 1 In Figure 2, basins of attraction of SH4, MSH4, K4, J4, HKT8, N8, WL8, SA8, RM8, and SK8 are shown for p_3. Based on the plots, our analysis indicates that SA8 and SK8 emerge as the most efficient and fast methods as they have wide regions of convergence and have brighter colors.

Example 3. Now, we consider a quartic polynomial,

$$p_4(z) = z^4 - 1,$$

with four roots +1, -1, i, and -i. In Figures 3, basins of attraction of SH4, MSH4, K4, J4, HKT8, N8, WL8, SA8, RM8, and SK8 are shown for p_4. From the plots, we observe that the methods N8 and SA8 have wider and brighter areas of convergence, thus, we conclude that for p_4, N8 and SA8 are the most effective methods.

Example 4. We now take into account a fifth degree polynomial as follows:

$$p_5(z) = z^5 - 1.$$

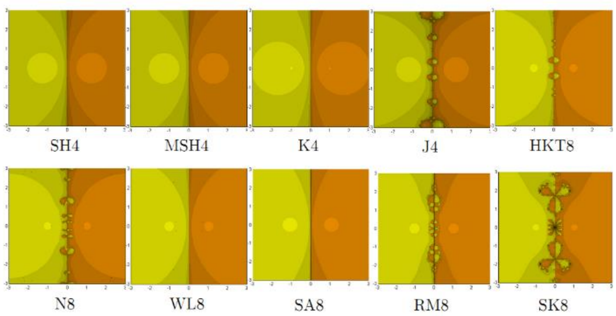
The zeros of the above polynomial are +1, 0.309+0.9511i, 0.309-0.9511i, -0.809+0.5878i, and -0.809-0.5878i. In Figure 4, basins of attraction of SH4, MSH4, K4, J4, HKT8, N8, WL8, SA8, RM8, and SK8 are shown for p_5. From the plots in Figure 4, we illustrate that the basins of attraction of SA8 have brighter colors and wider regions of convergence in comparison with the other methods under consideration. Hence, it is deduced that SA8 is the most effective method.

Example 5. Now we take into account the following polynomial with complex coefficients with roots, $-1+2i, -\frac{1}{2}-\frac{i}{2}, i, -\frac{3i}{2}, 1, 1-i$:

$$p_6(z) = z^6 - \frac{1}{2}z^5 + \frac{11}{4}(1+i)z^4 - \frac{1}{4}(19+3i)z^3 + \frac{1}{4}(11+5i)z^2 - \frac{1}{4}(11+i)z + \frac{3}{2} - 3i.$$

In Figure 5, basins of attraction of SH4, MSH4, K4, J4, HKT8, N8, WL8, SA8, RM8, and SK8 are shown for p_6. Based on the plots for p_6, we infer that SA8 is most efficient iterative approach among the other methods under consideration as it has wide regions of convergence and brighter colors.

Figure 1: Dynamical comparison of different iterative



methods for $p_2(z) = z^2 - 1$.

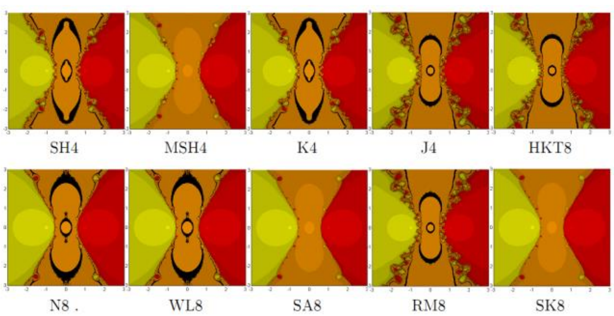


Figure 2: Dynamical comparison of different iterative methods for $p_3(z) = z^3 - z$.

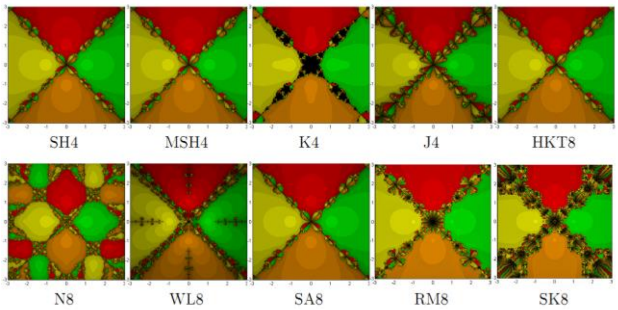


Figure 3: Dynamical comparison of different iterative methods for $p_4(z) = z^4 - 1$.

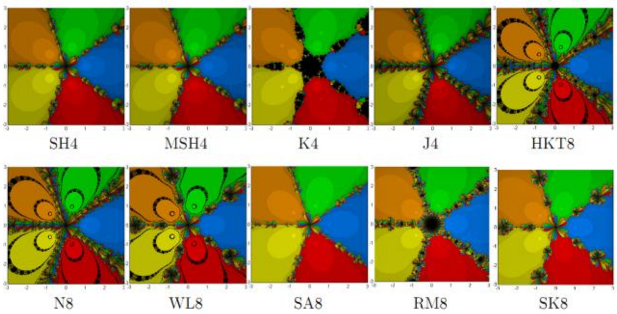


Figure 4: Dynamical comparison of different iterative methods for $p_5(z) = z^5 - 1$.

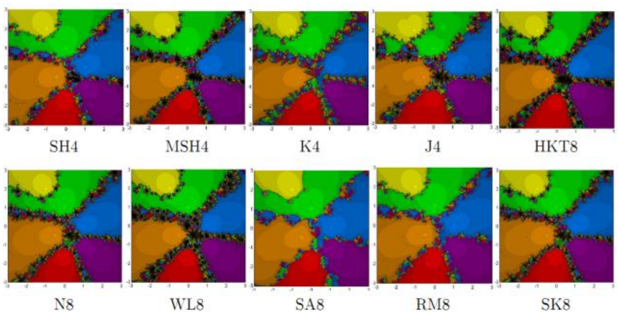


Figure 5: Dynamical comparison of different iterative methods for $p_6(z)$.

4. CONCLUDING REMARKS

In this study, we have conducted a graphical comparison of several optimal iterative methods of order four and eight. The dynamical analysis using basins of attractions illustrates that the optimal eighth-order scheme due to Sharma and Arora (SA8) is the best iterative method as it establishes simple basins of attraction and wider regions of convergence for both lower and higher degree polynomials when compared to similar optimal iterative methods.

DECLARATIONS

Conflicts of interest/Competing interests: The authors declare no any conflict of interest/competing interests.

Authors' contributions:

"**Moin-ud-Din Junjua:** Conceptualization, Supervision, Methodology, Writing- Original draft preparation **Shazia Altaf**: Formal analysis, Visualization, Validation **Fiza Sani**: Writing- Original draft preparation, Investigation, Software. ***Saima Akram***: Supervision, Software, Writing- Reviewing and Editing,"

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